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HARISH-CHANDRA DECOMPOSITION FOR ZONAL SPHERICAL FUNCTION OF TYPE A_n

A. Kazarnovski-Krol

Abstract. This paper is devoted to homological treatment of Harish-Chandra decomposition for zonal spherical functions of type A_n.

0. Introduction

Heckman-Opdam system of differential equations is holonomic , with regular singularities and has locally $|W|$-dimensional space of solutions (cf. corollary 3.9 of [12]), where $|W|$ is the cardinality of the Weyl group W. The system is a generalization of radial parts of Laplace-Casimir operators on symmetric Riemannian spaces of nonpositive curvature and is isomorphic to Calogero-Sutherland model in the integrable systems.

Harish-Chandra asymptotic solution is a unique solution of the system with the prescribed asymptotic behavior:

$$F_w(z) = z^{w\lambda+\rho}(1+\dots)$$

$(0 < |z_1| < |z_2| < \dots < |z_{n+1}|)$. Here $w \in W$ are elements of the Weyl group. These solutions provide a basis in the space of all the solutions in the chamber $0 < |z_1| < \dots < |z_{n+1}|$. Among all the solutions there is a distinguished one up to the constant multiplier, which admits continuation to analytic function at $z_1 = z_2 = z_3 = \dots = z_{n+1} \neq 0$. This solution is referred to as zonal spherical function. Zonal spherical function is normalized s.t. it is equal to 1 at $z_1 = z_2 = \dots = z_{n+1} = 1$.

Representation of the zonal spherical function as linear combination of elements of the basis (Harish-Chandra asymptotic solutions) is called Harish-Chandra decomposition.

In ref. [11] we provided an integral representation for the solutions of Heckman-Opdam system of differential equations in the case of A_n. We also described contours for integration Δ_w, integrals over them provide Harish-Chandra asymptotic solution $F_w(z)$. In ref. [34] we studied

the cycle $\mathbf{\Delta}$ for integration for zonal spherical function . This paper is devoted to homological treatment of Harish-Chandra decomposition for zonal spherical functions of type A_n. Namely, we explicitly decompose the distinguished cycle $\mathbf{\Delta}$ into linear combination of cycles Δ_w described in [11] and check that after normalization this turns out to be the Harish-Chandra decomposition for zonal spherical function of type A_n (theorem 2.2 and 3.1 below). The point of view that linear relations between the solutions reflect the linear relations in homology group is due to B. Riemann. He also emphasized the importance of the monodromy. In this case the corresponding homology theory is described in [2 , 37]. Harish-Chandra asymptotic solutions correspond to conformal blocks in conformal field theory (WA_n-algebras) and provide a basis in the space of conformal blocks, zonal spherical function is a particular conformal block, in the case of A_2 see figs. 3a,3b, 3c,3d,3e,3f and 2 below.

0.1 Notations.

$\alpha_1, \alpha_2, \dots, \alpha_n$ - simple roots of root system of type A_n

$\Lambda_1, \Lambda_2, \dots, \Lambda_n$- fundamental weights

R_+ - set of positive roots

$\delta = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ -halfsum of positive roots

k- complex parameter ('halfmultiplicity' of a root)

$$\rho = \frac{k}{2}\sum_{\alpha\in R_+}\alpha$$

$c(\lambda, k)$- c-function of Harish-Chandra

1. Multivalued form

Consider the following set of variables:

$z_l,\ \ l = 1, \dots, n+1,\ t_{ij},\ i = 1, \dots, j,\ \ j = 1, \dots, n.$

Variables z_l have meaning of arguments, while variables t_{ij} are variables of integration.

It is convenient to organize variables $z_l,\ t_{ij}$ in the form of a pattern, cf. fig 1.

The idea of such an organization is borrowed from Gelfand-Zetlin patterns [1].

Definition 1.1. Consider the following multivalued form $\omega(z, t)$:

$$
\begin{array}{ccccccccc}
z_1 & & z_2 & & \dots & & \dots & & z_{n+1} \\
& t_{1,n} & & t_{2,n} & & \dots & & t_{n,n} & \\
& & \dots & & \dots & & \dots & & \\
& & & t_{1,2} & & t_{2,2} & & & \\
& & & & t_{1,1} & & & &
\end{array}
$$

FIGURE 1. Variables organized in a pattern

$$
\begin{aligned}
\omega(z,t) := & \prod_{i=1}^{n+1} z_i^{\lambda_1+\frac{kn}{2}} \prod_{i_1>i_2} (z_{i_1}-z_{i_2})^{1-2k} \\
& \times \prod_{l=1}^{n+1}\prod_{i=1}^{n} (z_l - t_{i,n})^{k-1} \\
& \times \prod_{j=1}^{n-1}\prod_{i_1=1}^{j+1}\prod_{i=1}^{j} (t_{ij}-t_{i_1,j+1})^{k-1} \\
& \times \prod_{j=2}^{n}\prod_{i_1>i_2} (t_{i_1,j}-t_{i_2,j})^{2-2k} \\
& \times \prod_{j=1}^{n}\prod_{i=1}^{j} t_{ij}^{\lambda_{n-j+2}-\lambda_{n-j+1}-k} \quad dt_{11}dt_{12}dt_{22}\dots dt_{nn}
\end{aligned}
$$

Remark 1.2. k is a complex parameter - 'halfmultiplicity' of a restricted root, cf. Heckman, Opdam [12].

In [11] we proved that integrals over the form $\omega(z,t)$ over appropriate cycles provide all the solutions to Heckman-Opdam system of differental equtions and described cycles Δ_w for Harish-Chandra asymptotic solutions (definition 4.3 and theorem 6.3 of [11]).

Definition 1.3. A complex number z can be represented as $z = re^{i\alpha}$, where r, α are real numbers, $r \geq 0$. r is called absolute value of z, while α is called the phase of z. When we say that the phase of a complex number z is equal to 0, we mean that $\alpha = 0$, or the number itself is real and nonnegative.

2. The distinguished cycle $\mathbf{\Delta}$

Assume that $z_1, z_2, \ldots, z_{n+1}$ are real and

$$0 < z_1 < z_2 < \ldots < z_{n+1}.$$

Definition 2.1. Define cycle $\mathbf{\Delta} = \mathbf{\Delta}(z)$ by the following inequalities:
$t_{i,j+1} \le t_{ij} \le t_{i+1,j+1}$ and
$z_i \le t_{in} \le z_{i+1}$.
Define form $\omega_\Delta(z,t)$ as:

$$\begin{aligned}
\omega_\Delta(z,t) := & \prod_{i=1}^{n+1} z_i^{\lambda_1 + \frac{kn}{2}} \prod_{i_1 > i_2} (z_{i_1} - z_{i_2})^{1-2k} \\
& \times \prod_{i \le l} (z_l - t_{i,n})^{k-1} \prod_{i > l} (t_{i,n} - z_l)^{k-1} \\
& \times \prod_{j=1}^{n-1} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j+1})^{k-1} \prod_{i_2 \ge i_1} (t_{i_2,j+1} - t_{i_1,j})^{k-1} \\
& \times \prod_{j=2}^{n} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j})^{2-2k} \\
& \times \prod_{j=1}^{n} \prod_{i=1}^{j} t_{ij}^{\lambda_{n-j+2} - \lambda_{n-j+1} - k} \quad dt_{11} dt_{12} dt_{22} \ldots dt_{nn}
\end{aligned}$$

It is assumed that phases of factors in the formula for $\omega_\Delta(z,t)$ are equal to zero if k and $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are real. For the homological meaning of the cycle $\mathbf{\Delta}$ see fig. 2 below and theorem 5.7 of [34].

In [11] cycles $\Delta_w(z)$ and forms $\omega_w(z,t)$ were described,in the case of A_2 see figs. 3a,3b,3c,3d,3e,3f below. In the case A_n the figures are similar.

FIGURE 2. Zonal spherical function

FIGURE 3A

The following theorem explains the relation between $\Delta(z)$ and $\Delta_w(z)$.

Theorem 2.2. *(Harish-Chandra decomposition)*

$$\int_{\Delta(z)} \omega_{\Delta(z,t)} = \sum_{w \in S_{n+1}} b(w, \lambda, k) \int_{\Delta_w(z)} \omega_w(z),$$

where

FIGURE 3B

$$b(w,\lambda,k) = \frac{e^{2\pi i(\lambda,\delta)}e^{\pi i \ l(w)(k-1)}}{(2i)^{\frac{n(n+1)}{2}} \prod\limits_{\alpha\in R_+} \sin(-\pi(w\lambda,\alpha^\vee))}$$

The theorem is an application of the two following lemmas, also section 2 of [11] is useful.

Lemma 2.3. *(Elementary decomposition) Let $z_1, z_2, \ldots, z_n$ be real and $0 < z_1 < z_2 < \ldots < z_n$. Consider the following integral:*

$$\int t^{a_0-1}(z_1-t)^{a_1-1}(z_2-t)^{a_2-1}\ldots(z_n-t)^{a_n-1}dt$$

and consider contours $\gamma_1(t), \gamma_2(t), \gamma(t),\ t \in [0,1]$ as follows.

$$\gamma(t) = tz_{i-1} + (1-t)z_i$$

$\gamma_1(t)$ is a loop which starts and ends at z_{i-1} and goes counterclockwise s.t. the following inequalities are fullfilled for all $t \in [0,1]$:

$$z_{i-2} < |\gamma_1(t)| \le z_{i-1}$$

FIGURE 3C

$\gamma_2(t)$ is a loop which starts and ends at z_i and goes counterclockwise ,s.t. the following inequalities are fullfilled for all $t \in [0,1]$:

$$z_{i-1} < |\gamma_2(t)| \leq z_i$$

as indicated on fig. 4, phases of the factors should be appropriately chosen. The following is the specific choice of the phases : if all $a_0, a_1, \ldots, a_n$ are real, then the phase of the integrand along $\gamma_1(t), \gamma_2(t), \gamma(t)$ is chosen to be zero for small values of t. Then we have the following relation between γ_1, γ_2 and γ:

$$\gamma = -\gamma_1 \times \frac{e^{-\pi i(a_0+a_1+\ldots+a_{i-1})}}{(2i)\sin\pi(a_0+a_1+\ldots+a_i)} + \gamma_2 \times \frac{e^{-\pi i(a_0+a_1+\ldots+a_i)}}{(2i)\sin\pi(a_0+a_1+\ldots+a_i)}$$

FIGURE 3D

Lemma 2.4. *(Elimination of 'wrong' diagrams). Integrals of the form* $\omega(z,t)$ $(\omega_w(z,t))$*, such that contours for integration of* t_{ij}, $t_{i,j+1}$, $t_{i+1,j+1}$ *are shown on fig. 5, provided* k *is not an integer, are equal to zero. We suppose that* t_{ij} *goes from* $t_{i,j+1}$ *to* $t_{i+1,j+1}$ *,* $t_{i,j+1}$ *goes from* $t_{i+1,j+2}$ *to* $t_{i+1,j+2}$*, and* $t_{i+1,j+1}$ *goes from* $t_{i+1,j+2}$ *to* $t_{i+1,j+2}$ *cf. fig. 5a. The same holds true for* $t_{i-1,n-1}, t_{i-1,n}, t_{i,n}$*, and* z_i *correspondingly, cf. fig. 5b.*

By the 'wrong' diagrams we mean diagrams ,where the two arrows have the same target, see figs. 5 and 6 of [34] .

Figure 3e

Remark 2.5. Lemma 2.4 is equivalent to quantum Serre's relations in the form given in [3], see also [2].

3. Normalization

Let

$$F_w(z) = \Big(\prod_{\alpha \in R_+} \frac{\Gamma((-w\lambda, \alpha^\vee)) \sin(\pi(-w\lambda, \alpha^\vee))}{\Gamma((-w\lambda, \alpha^\vee) + k)}$$
$$\times e^{-2\pi i(\lambda, \delta)} e^{-\pi i(k-1)l(w)} \Gamma(k)^{\frac{n(n+1)}{2}} (2i)^{\frac{n(n+1)}{2}} \Big)^{-1} \int_{\Delta_w(z)} \omega_w(z, t)$$

Then

$$F_w(z) = z^{w\lambda + \rho}(1 + \dots)$$

cf. [11] theorem 6.1.

Also, let

$$F_\Delta(z) = \frac{\Gamma(k)\Gamma(2k)\dots\Gamma((n+1)k)}{\Gamma(k)^{\frac{(n+1)(n+2)}{2}}} \int_{\Delta(z)} \omega_\Delta(z, t)$$

Figure 3f

Then $F_\Delta(1, 1, \dots, 1) = 1$, cf. [10] theorem 1.5.

After this normalization theorem 1 reads as usual Harish-Chandra decomposition cf. [12,15].

Theorem 3.1. *In the above normalization we have:*

$$F_\Delta(z) = \sum_{w \in S_{n+1}} c(w\lambda, k) F_w(z),$$

where $c(w\lambda, k)$ is a c-function of Harish-Chandra:

FIGURE 4. Elementary decomposition.

FIGURE 5A. Cycles of this type are homological to zero

$$c(w\lambda, k) = \frac{\prod_{\alpha\in R_+} \frac{\Gamma((\rho,\alpha^\vee)+k)}{\Gamma((\rho,\alpha^\vee))}}{\prod_{\alpha\in R_+} \frac{\Gamma((-w\lambda,\alpha^\vee)+k)}{\Gamma((-w\lambda,\alpha^\vee))}}$$

I.e. $F_\Delta(z)$ *is identified with zonal spherical function.*

FIGURE 5B. Cycles of this type are homological to zero

Corollary 3.2. *Suppose* $z_1(t), z_2(t), \ldots, z_{n+1}(t)$*,* $t \in [0,1]$ *are closed loops on a complex plane, i.e.* $z_1(0) = z_1(1), z_2(0) = z_2(1), \ldots, z_{n+1}(0) = z_{n+1}(1)$*, such that* $z_i(t) \neq z_j(t)$ *for* $i \neq j$*. Let also* $Re(z_i(t)) > 0$ *for each* $i = 1, \ldots, n+1$*. Then the homological class of the cycle* $\mathbf{\Delta}$ *is preserved under the monodromy along paths* $z_i(t)$*.*

Remark 3.3. In this approach multiplicative structure of c-function of Harish-Chandra gets a very simple explanation. Namely:

$$c(\lambda, k) = \frac{\prod_{1 \leq i < j \leq n} \frac{\Gamma((\rho, e_i - e_j) + k)}{\Gamma((\rho, e_i - e_j))}}{\prod_{1 \leq i < j \leq n} \frac{\Gamma((-w\lambda, e_i - e_j) + k)}{\Gamma((-w\lambda, e_i - e_j))}} \times \frac{\prod_{1 \leq i < n+1} \frac{\Gamma((\rho, e_i - e_{n+1}) + k)}{\Gamma((\rho, e_i - e_{n+1}))}}{\prod_{1 \leq i < n+1} \frac{\Gamma((-w\lambda, e_i - e_{n+1}) + k)}{\Gamma((-w\lambda, e_i - e_{n+1}))}}$$

Here $\{e_i - e_j | \quad 1 \leq i < j \leq n+1\}$ are positive roots of root system of type A_n. Multiplicative properties of c-function of Harish-Chandra were observed by Bhanu-Murti in the case of $SL(n, \mathbb{R})$ and in general case by Gindikin and Karpelevich [17]. c-function of Harish-Chandra is equal to the product of elements of $6j$-symbols , see fig. 6. Multilpicative structure of c-function of Harish-Chandra amounts to simple combinatorics related to positive roots , in this case:

$$\{e_i - e_j | 1 \leq i < j \leq n+1\} = \{e_i - e_j | 1 \leq i < j \leq n\} \bigcup \{e_i - e_{n+1} | 1 \leq i < n\}.$$

FIGURE 6 . c-function of Harish-Chandra as a product of elements of $6j$- symbols.

This combinatorics is both very instructive and restrictive.

Remark 3.4. We have also checked the monodromy properties of the cycle $\mathbf{\Delta}$ using quantum group argument, see [34].

Remark 3.5. Harish-Chandra decomposition for zonal spherical function might be considered as an analogue of Bernstein-Gelfand-Gelfand resolution.

Concluding remark. We would like to point out once more that the distinguished cycle $\mathbf{\Delta}$ appeared in the classical calculation of Gelfand and Naimark [16] of zonal spherical function for $SL(n, \mathbb{C})$, originates in the so-called elliptic coordinates and provides a materialization of the flag manifold.

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Department of Mathematics Rutgers University New Brunswick, NJ 08854, USA